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NOTES ON SEARCH, DETECTION
AND LOCALIZATION MODELING

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Preface

This report is a collection of material that has been used in courses on search, detection and localization modeling. Its organization follows to some extent material by S. M. Pollock in Selected Methods and Models in Military Operations Research which is listed in the report bibliography. The report is not intended to be a text on these subjects. In particular, in some areas it does not provide the depth of coverage that is found in the book Search and Detection by Alan R. Washburn which is cited in this report as Reference 22.

In the third revision, typographical and other errors have been corrected and, in addition, changes and additions have been made to Sections XII and XIV of the report.

Table of Contents

I. Detection Models and Signal Detection Theory

Signal detection theory is the basis for analyzing the detection models that are described in this report. In signal detection theory, the decision making portion of a detection system is called the receiver and a detection experiment is the observation by a receiver of input data accumulated during some time interval. The data that is related to a target is called signal. The data that is not related to the target is called noise. In general, the target data is associated with a localization region that in some cases is called a resolution cell. When a detection experiment is performed, either the event H_0 = (the receiver input is noise) or its complement H_1 = (the receiver input is signal and noise) will occur. In the first detection models that are described here, after analysis of the input data by a receiver, either the event D_0 =(the receiver decides the input is noise) or its complement D_1 = (the receiver decides the input is signal and noise) will also occur. Detection models for which D_1 is the complement of D_0 are called binary detection models or forced choice detection models. Four events which are important in binary detection models are indicated in the Venn diagram of Figure 1.

The Venn diagram emphasizes a decision problem that is associated with a receiver that can be modeled using a binary detection model. The problem is this: Under what conditions should the event D_1 occur? That is, under what conditions

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should a receiver decide that the input data accumulated during the observation time interval is signal and noise?

	H_0	H_1
D_0	$D_0 \cap H_0$	$D_0 \cap H_1$
D_1	$D_1 \cap H_0$	$D_1 \cap H_1$

Figure 1. Four events of importance in binary detection models.

In the detection model descriptions that follow, the following notation and terminology are used: $P_f = P(D_1|H_0)$, the probability of D_1 given H_0, is called the **false alarm probability**; $P_d = P(D_1|H_1)$, the probability of D_1 given H_1, is called the **detection probability** and $P = P(H_1)$, the probability of H_1, is called the **prior probability**.

In the detection models, the input to a receiver is determined by a stochastic process that has the following characteristics: It is a random noise proce s when there is no target data and it is a random noise process plus a signal process when there is target data. Although the receiver input process in some cases might appear to be determined by a continuous parameter stochastic process, because of the finite amount of information (unique data) contained in a bounded sequence of finite length, a discrete parameter stochastic

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process is sufficient to determine the receiver input in these cases. This is established formally by the stochastic sampling theorem. Consequently, in these models, the input to a receiver is determined by a sequence of random variables $Y_1, \cdots , Y_m$ and an observation yields a sequence of values $y_1, \ldots y_m$.

Three detection models are described in Section III. In the first model, the signal process is a deterministic process. In the second and third, the signal process is a random process. To define a random noise process or a random signal process, only the joint distribution of the finite sequence of random variables that determine the process needs to be specified. If the signal process is a deterministic process, the signal values can be determined before an observation is performed. To define the process in this case, only these values need to be specified.

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II. Decision Criteria

To simplify the discussion of decision criteria and
decision rules, a receiver's input will be assumed to be
determined by a single decision random variable Y. In this
case, the input process in determined by the conditional
distribution function $F_Y(y|H_0)$ when the input is noise alone
and by the conditional distribution function $F_Y(y|H_1)$ when the
input is signal plus noise.

The condition that a receiver's input is required to
satisfy in order that the event D_1 will occur can be specified
in terms of a decision rule. For the assumed case, a decision
rule is a rule which determines for every observable value of Y
the decision that the receiver is to make. The decision rule can
be considered to be a function $\phi(y)$ which relates each
observable value y to one or the other of the following two
commands: d_0 = "decide that the receiver input was noise" and
d_1 = "decide that the receiver input was signal and noise".
Choosing a decision rule $\phi(y)$ defines a set Ω of observable
values of Y such that the event $D_1 = \{ Y \varepsilon \Omega \}$.

The problem which was considered in Section I can now be
restated in the following way: What criterion should be adopted
in order to determine a decision rule or, equivalently, its
corresponding set Ω ? A desirable characteristic for a criterion
is suggested by the following argument: Consider the odds in
favor of H_1 given y is observed. That is, consider

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$P(H_1|Y = y)/P(H_0|Y = y)$. One might expect that y would be a member of the set Ω if and only if y made this ratio equal to or greater than some value k. But this is equivalent to defining Ω as follows: $\Omega = \{ y : L(y) \geq K \}$ where $L(y)$ is the likelihood ratio associated with an observed value y and K is a constant related to the constant k. This suggests that choosing an optimum criterion is equivalent to choosing an optimum value for K. Four specific decision criteria are defined next in terms of K. For each criterion, Ω has the above form. But for each criterion the choice of K is different. The decision criteria are:

1. **The Neyman-Pearson Criterion:** Choose Ω so that p_d is a maximum subject to the constraint that $p_f \leq \alpha$ where α is a specified value. For a continuous decision random variable, the constant K is chosen so that $p_f = \alpha$.

2. **The Bayes Criterion:** Choose Ω so that the expected cost of a receiver's decision is a minimum. For a continuous decision random variable, $K = [(c_{10}-c_{00})/(c_{01}-c_{11})] \cdot (1-P)/P$ if $c_{10} > c_{00}$ and $c_{01} > c_{11}$ where c_{ij} is the cost of $D_i \cap H_j$.

3. **The Ideal Observer Criterion:** Choose Ω so that the probability that the receiver makes an incorrect decision is a minimum. $K = (1-P)/P$ for a continuous decision random variable.

4. **The Minimax Criterion:** Choose Ω when P is unknown so that the maximum expected cost of a receiver's decision is a minimum. If $c_{10} > c_{00}$ and $c_{01} > c_{11}$, then

$K = [(c_{10}-c_{00})/(c_{01}-c_{11})] \cdot (1-P^*)/P^*$ for a continuous decision random variable. Here, P^* is the value of the prior probability P that would make the expected cost of a receiver's decision a maximum if P were known and the Bayes Criterion were used.

If a model which specifies the conditional distributions $F_Y(y|H_0)$ and $F_Y(y|H_1)$ and a decision rule are adopted, then the value of p_f and the value of p_d are determined. This pair of values (p_f, p_d) is called a receiver operating point. If the decision rule results from using a likelihood ratio criterion such as one of the four listed above, then it will involve the parameter K since $\Omega = \{y: L(y) \geq K\}$. And, for a given value of K, since Ω uniquely determines the pair (p_f, p_d), a single operating point results. By varying K, a set of operating points can be generated which determines a receiver operating characteristic curve or ROC curve. Different ROC curves can be produced by changing either one or both of the conditional distributions which implies either a change in the signal process or a change in the noise process.

A decision rule which results from using a likelihood ratio criterion in a model in which the input process is determined by a set of m random variables can be expressed in terms of a set Ω as follows: $\Omega = \{ (y_1, \cdots, y_m) : L(y_1, \cdots, y_m) \geq K \}$ where K is specified in the same way that it is when $m = 1$.

III. Three Binary Detection Models

Three detection models are examined in this section.
For the first two detection models, the input stochastic process
for an observation is defined by a time sequence of continuous
random variables. The random variables represent a sample from a
continuous parameter stochastic process which is sampled at times
such that the random variables are independent. For the third
detection model, the input stochastic process is a counting
process and it is defined by a single discrete random variable
that is equal to the number of events that are counted during the
observation.

Model I: In the first detection model, a sampled noise
value is a value of a normally distributed random variable with
mean zero and with known variance σ^2. And a sampled signal
value is a known value of a deterministic variable. Thus, the
input process corresponding to an observation consists of some
number m of independent normal random variables $Y_1, \cdots , Y_m$
each with variance σ'. And, for $i = 1,2, \cdots ,m$, when a signal
is not present the mean of Y_i is zero and when a signal is
present the mean is s_i. The result of using a likelihood ratio
decision rule in the model can be expressed in terms of a random
variable Z. This random variable is called a crosscorrelation
statistic and it is defined by $Z = \Sigma \, s_i \cdot Y_i$ where the sum index
$i = 1,2, \cdots ,m$ here and in the remainder of this section.
However, it is more convenient to express the result in terms of
a statistic V which is defined by $V = Z/\sigma_2$. In terms of this

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random variable, the conditional probabilities p_f and p_d are given by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$ where Φ symbolizes the standard normal cumulative distribution function, $v^* = (1/\sigma_z) \cdot (\sigma' \ln K + (1/2) \Sigma s_i')$, is determined by the decision rule and $d = \Sigma s_i'/\sigma'$ is called the detection index.

Often, the input stochastic process represents a quantity whose square is proportional to power. In such a case, the average receiver input power is the random variable $\Sigma Y_i'/m$. If a signal is not present, the expected average receiver input power is $N = \Sigma \sigma'/m = \sigma'$ where N is called the noise. The average receiver input signal power is $S = \Sigma s_i'/m$ where S is called the signal. In these terms, $d = m \cdot (S/N)$ where S/N is called the signal-to-noise ratio.

If a receiver's input can be considered to be a time sequence of continuous voltage values such as in the case of a sonar receiver, in some cases a frequency representation can be used that involves the concept of receiver bandwidth. In these cases, the noise process is assumed to be such that $m = t/\delta t$ where t is the integration time (the duration of an observation) and δt is the time between samples with $\delta t = 1/[2(BW)]$ where BW is the bandwidth and δt is determined by the sampling theorem. This implies that the detection index can be written as $d = 2t \cdot (BW) \cdot (S/N)$. By defining N_0 as the power spectral density where $N_0 = N/BW$, the detection index can also be written as $d = 2t \cdot (S/N_0)$.

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In Reference 2, the conditions required by the first model are called Case I and in the following sections the first model is called the Case I model. A receiver that processes data such that it would implement a likelihood ratio decision rule under the conditions of the first model is called a matched filter or crosscorrelation detector. If the description of the input noise is adequate, a Case I model can be used to obtain an estimate of an upper bound on a detection system's performance, since all the information necessary to define the signal is assumed to be known.

Model II: In the second detection model, a sampled noise value is a value of an independent normal random variable with mean zero and with known variance σ'. And a sampled signal value is an independent random variable with mean zero and with known variance σ_s^2. Thus, the input process corresponding to an observation consists of some number m of independent normal random variables $Y_1, \cdots, Y_m$ each with mean zero and each with variance σ' when a signal is not present and each with variance $\sigma' + \sigma_s^2$ when a signal is present. The result of applying a likelihood ratio decision rule in this model can be expressed in terms of a statistic X which is defined by $X = \Sigma Y_i^2$.

When a signal is not present, the statistic X/N has a chi-square distribution with m   ·es of freedom. When a signal is present, the statistic $\not\vdash \vee \vec{\jmath}$) has a chi-square distribution with m degrees of freedom. So, in terms of these

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two statistics, the two conditional probabilities p_f and p_d are: $p_f = P(X_m^2 \geq x^*/N)$ and $p_d = P(X_m^2 \geq (x^*/N) \cdot [1/(1+S/N)])$ where X_m^2 is a chi-square random variable with m degrees of freedom, x^* is a number which is determined by the decision rule and S/N is the signal-to-noise ratio. A receiver that would implement a likelihood ratio decision rule under the conditions of the second model is called an energy detector or square law detector.

The mean of a chi-square random variable with m degrees of freedom is m and the variance is $2m$. By the central limit theorem, as the number of degrees of freedom of a chi-square random variable becomes large, it can be approximated by a normal random variable that has the same mean and variance. Therefore, p_f and p_d can be approximated by $p_f = 1 - \Phi[(x^*/N - m)/(2m)^{\frac{1}{2}}]$ and $p_d = 1 - \Phi\{[1/(1+S/N)][x^*/N - m - m \cdot (S/N)]/(2m)^{\frac{1}{2}}\}$. And, with $v^* = [x^*/N - m)/(2m)^{\frac{1}{2}}]$ and $d = (m/2) \cdot (S/N)^2$, this becomes: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1+S/N)] \cdot (v^* - d^{\frac{1}{2}})\}$.

Based on the above approximation, if the noise N is significantly larger than the signal S, then p_f and p_d can be approximated by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$. And, if the concept of bandwidth is applicable so that the sample size $m = 2t \cdot (BW)$, then $d = t \cdot (BW)(S/N)^2$. In Reference 2, the conditions required for this approximation are called Case II and in the following sections this limiting form of the second model is called the Case II model.

Model III: In the third detection model, a sampled noise value and a sampled signal value are values of independent random variables that are determined b' independent Poisson processes that ar(observed for a time interval t. The noise process is characterized by a counting rate α, the signal process is characterized by a counting rate α_s and the noise and signal processes are additive. This implies that when the input is noise alone, the input is a Poisson random variable with parameter $\alpha \cdot t$, the expected number of noise counts, and when the input is signal and noise, the input is a Poisson random variable with parameter $(\alpha + \alpha_s) \cdot t$, the expected number of noise and signal counts.

For a likelihood ratio decision rule, $p_f = 1 - P(y^*; \iota \cdot t)$ and $p_d = 1 - P[y^*; (\alpha + \alpha_s) \cdot t]$ where y^* is a threshold value that is determined by the decision rule and $P(y; \theta)$ represents the Poisson cumulative distribution function with parameter θ. When θ is large, the cumulative distribution function can be approximated by the cumulative distribution function of a normal random variable that has the same mean and variance. Using this approximation for cases where $\alpha \cdot t$ is sufficiently large, since both the mean and variance of a Poisson random variable are equal to θ, $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi\{[1/(1 + \alpha_s/\alpha)^{\frac{1}{2}}](v^* - d^{\frac{1}{2}})\}$ where $v^* = (y^* - \alpha t)/(\alpha t)^{\frac{1}{2}}$ and $d = \alpha \cdot t \cdot (\alpha_s/\alpha)'$. If, in addition, α is significantly larger than α_s, that is, if $\alpha_s/\alpha \ll 1$ as well as $\alpha \cdot t \gg 1$, then p_f and p_d can be approximated by: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$.

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The third detection model might be used to describe a
receiver whose input for an observation is the number of photons
counted by a radiation detector in situations where $\alpha \cdot t$, the
expected number of counts when no signal is present, is of the
order of thirty or more.

When a likelihood ratio decision rule is used in the three
models discussed above, for the first model and under limiting
conditions for the second and third models, the following result
is obtained: $p_f = 1 - \Phi(v^*)$ and $p_d = 1 - \Phi(v^* - d^{\frac{1}{2}})$ where the
value of v^* depends on the noise power N for the first and
second models. For a sonar receiver described by the first
model, that is, by the Case I model: $d = 2t \cdot (BW)(S/N)$. For a
sonar receiver described under the limiting conditions for the
second model, that is, by the Case II model, $d = t \cdot (BW)(S/N)^2$.
So, in either a Case I model or a Case II model of a sonar
receiver, the detection index d is a function of the time
bandwidth product $t \cdot (BW)$ and the signal-to-noise ratio S/N.
Since sonar equations relate S/N to system, target and
environmental parameters, a sonar equation can be used to relate
S/N to these parameters in a model of a sonar receiver.

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IV. General Detection Models

The detection models that have been considered to this
point are based on binary detection theory. After each
observation, a receiver decides either that the input
corresponding to the observation was noise or else it decides it
was signal plus noise. However, in some detection systems this
decision is delayed. In a computational sense, a model of such a
detection system is generally complex relative to a binary
detection model. To illustrate this, consider an active sonar
system whose receiver includes an operator. Suppose the
probability that the operator will detect a target echo has been
determined in a laboratory experiment in which the operator was
required to decide after each input corresponding to a resolution
cell that either the input was a target echo (signal) and noise
or the input was noise alone. In addition, suppose that under
operational conditions the operator normally delays this
decision. Then, in general, the probability that the operator
will decide that the input corresponding to a resolution cell
that contains a target is a target echo and noise will not be
equal to the probability of the event in the forced choice
experiment. And, in addition, the probability that the operator
will decide the input corresponding to a resolution cell that
does not contain a target is a target echo and noise will not be
equal to the probability of this event in the forced choice
experiment. Consequently, in general, the value of both p_d and

p_f for an operational environment will be different than that for the laboratory environment.

One model that has been proposed to deal with this kind of situation defines the event that a receiver decides that the input corresponding to a resolution cell is signal and noise to be equivalent to the event that out of n consecutive observations at least k of them would result in the decision that the input was signal and noise in a forced choice experiment. The model is said to be based on an k-out-of-n detection criterion. With this criterion, the probability that a target will be first detected on the jth observation can be found as follows: Determine the 2^j sequences of forced choice responses that could result for a sequence of j consecutive observations. Next, determine the probability of occurrence for each sequence that first satisfies the k-out-of-n detection criterion on the jth observation. The probability of first detection on the jth observation is equal to the sum of these probabilities. The cumulative probability of detection at the jth observation is the sum of the probabilities of first detection on the ith observation for $i = 1, 2, \cdots, j$.

V. Signal-to-Noise Ratio Detection Models

In some radar and sonar detection models, for a specified value of p_f, a minimum acceptable value of p_d is defined. This minimum acceptable value of p_d and the specified value of p_f define what can be called a minimum acceptable signal-to-noise ratio $(S/N)_m$ if p_d is a nondecreasing function of signal-to-noise ratio. In some sonar detection models, $(S/N)_m$ in decibels is called the **detection threshold** DT. In symbols, $DT = 10 \log(S/N)_m$. If the minimum acceptable value of p_d is .5, then DT is usually called the **recognition differential** RD. The difference between the signal-to-noise ratio in decibels and RD (or DT) is called the **signal excess** SE. In symbols, $SE = 10 \log(S/N) - RD$.

One interpretation of signal excess is that for a localization region containing a target detection occurs with probability one if $SE \geq 0$ and with probability zero if $SE < 0$. This interpretation provides the basis for defining detection in the three encounter detection models that are discussed in Section VII. A more consistent interpretation of signal excess is: If $SE \geq 0$, then the probability of detection p_d is greater than or equal to the minimum acceptable value (.5 if recognition differential RD is used to define signal excess). For cases where p_d increases rapidly with signal excess in the neighborhood of zero signal excess, the two interpretations may be operationally equivalent. For a discussion of this point as

well as a discussion of an operational case in which receiver decisions are delayed, see Reference 3.

Signal excess (signal-to-noise ratio) detection models provide a basis for general detection models, in particular, models that describe nonstationary noise and signal processes and randomly changing decision rules. This is illustrated by the models described in Section VII. In addition, signal excess models provide a basis for delayed receiver decision models. This is illustrated by the active sonar detection models in both Reference 4 and Reference 5 that are based on a k-out-of-n detection criterion. In all of these models, the signal-to-noise ratio and the recognition differential are random variables.

Using $X(t)$ to represent a random variable corresponding to an index time t and a subscript to identify the random variable in such models, for a passive sonar receiver, the signal-to-noise ratio in decibels associated with a decision at the index time is: $X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)]$. In this expression, SL represents source level, TL represents transmission loss, NL represents noise level and DI represents directivity index. Since signal excess SE is defined to be the difference in decibels between the signal-to-noise ratio and the recognition differential (or detection threshold), it too is a random variable and, for any decision time t, one can write:

(1) $X_{SE}(t) = X_{SL}(t) - X_{TL}(t) - [X_{NL}(t) - X_{DI}(t)] - X_{RD}(t)$.

The distributions of the random variables on the right side of Equation 1 determine the distribution of the signal excess. In

the passive sonar detection model described in Reference 6,
$X_{SL}(t)$, $X_{RD}(t)$ and, in effect, $X_{NL}(t)$ are normally
distributed random variables while $X_{TL}(t)$ is a uniformly
distributed random variable. In the three signal excess models
that are described in Section VII, all of the random variables in
Equation 1 are normally distributed.

It is sometimes convenient to write Equation 1 as follows:

(2) $X_{SE}(t) = SE(t) + X(t)$.

In Equation 2, $SE(t)$ is the expected value of the signal excess
determined by the following expected value equation:

(3) $SE(t) = SL(t) - TL(t) - [NL(t) - DI(t)] - RD(t)$

where each term on the right represents the expected value of the
indicated random variable and $X(t)$ is a random variable that
determines the stochastic character of the signal excess. Since
$SE(t)$ is the mean of $X_{SE}(t)$; by Equation 2, the mean of $X(t)$
is equal to zero and the standard deviation of $X(t)$ is equal to
the standard deviation of $X_{SE}(t)$. If σ represents the
standard deviation of $X_{SE}(t)$ and the random variables on the
right side of Equation 1 are statistically independent, then
$\sigma^2 = \sigma_{SL}^2 + \sigma_{TL}^2 + \sigma_{NL}^2 + \sigma_{DI}^2 + \sigma_{RD}^2$. This relation has been used to
determine a standard deviation for the signal excess in
operational models.

VI. General Encounter Models

A basic problem associated with search modeling is that of determining the probability that a target will be detected by a detection system during an encounter with one or more detection systems. In the encounter models that are considered in this report, during a search, observations are made of a series of localization regions. The probability of detection on an observation is $P(D_1 \cap H_1)$. The probability of a false alarm on an observation is $P(D_1 \cap H_0)$. In these models, the time to resolve a false alarm is ignored. However, p_d and p_f are assumed to be determined by some criterion such that p_f is an operationally reasonable value.

Using the order number of a decision rather than its time as an index relative to detection decisions for localization regions that contains a target and a random variable N to represent the decision order number at which detection first occurs, the probability of detection during an encounter can be written as:

$$P(N \leq n) = P(N \leq m) + P(N = m+1) + \cdots + P(N = n)$$ or equally as

$$P(N \leq n) = 1 - [1 - P(N \leq m)] \cdot (1 - g_{m+1}) \cdots (1 - g_n)$$ where

$g_i = P(N = i | \overline{N \leq i-1})$ is the probability of the event detection at the ith decision conditioned on the event no detection at an earlier decision and $1 \leq m \leq n$. The second expression is generally of greater interest than the first expression, since g_i can usually be more directly related to operational parameters such as range and environmental conditions that determine a target's detectability than can $P(N = i)$.

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With a time rather than the order number to index a decision and a random variable T to represent the time index at which detection first occurs, $P(N \leq n)$ becomes $P(T \leq t_n)$ with

$$P(T \leq t_n) = 1 - [1 - P(T \leq t_m)] \cdot [1 - g(t_{m+1})] \cdots [1 - g(t_n)]$$

where $g(t_i) = P(T = t_i | T \leq t_{i-1})$.

If $g(t_i) \ll 1$ for $i = 1, 2, \cdots, n$, then, to a first approximation, $\ln[1 - g(t_i)] = -g(t_i)$ for $i = 1, 2, \cdots, n$ and $P(T \leq t_n) = 1 - [1 - P(T \leq t_m)] \cdot \exp[-\Sigma\ g(t_i)]$. This follows since $P(T \leq t_n) = 1 - [1 - P(T \leq t_m)] \cdot \exp[\Sigma\ \ln[1 - g(t_i)]$ where the sum index $i = m+1, \cdots, n$. A continuous analog to this approximation can be used to describe an encounter for which $g(t_i) \ll 1$ for $i = m, m+1, \cdots, n$ and decisions during the encounter can be considered to occur continuously. That is, the time of an observation corresponding to a decision and the time between decisions are both negligible relative to the time of the encounter.

The analog can be developed as follows: First, let δt be the time between decisions, then $t_i = i \cdot \delta t$ and the probability of detection $P(T \leq t_n) = 1 - [1 - P(T \leq t_m)] \cdot \exp[-\Sigma\ \tau(t_i) \cdot \delta t]$ where $\tau(t_i) = (1/\delta t) \cdot g(t_i)$ is a detection rate function (a probability of detection per unit time) and, in terms of δt, the probability $g(t_i) = P[T = i \cdot \delta t | T \leq (i-1) \cdot \delta t]$.

If T is considered to be a continuous random variable, the expression for $P(T \leq t_i)$ above indicates that the sum in the exponent should be replaced by an integral whose integrand is a continuous function $\tau(t)$. If $\tau(t)$ can be determined, then,

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with $g(t_i)$ as a guide, the cumulative probability of detection $P(T \leq t)$ can be def ed by:

(4) $\qquad \tau(t) = \lim \{(1/\delta t)\ P(t < T \leq t+\delta t | \overline{T \leq t})\}$

where the limit is for δt approaching zero. Equation 4 implies the differential equation: $dp(t)/dt = [1 - p(t)] \cdot \tau(t)$ where $p(t) = P(T \leq t)$. A solution to this equation is:

(5) $\qquad P(T \leq tn) = 1 - [1 - \Gamma(T \leq tm)]\ \exp[-\int_{tm}^{tn} \tau(t)dt]$

where t is the time index for a decision during an encounter, tm is some time during the encounter and tn > tm. A $\tau(t)$ that is based on a visual detection model is described in Reference 7. If the detection capability of a detection system is assumed to depend on a target's position relative to the detection system during an encounter but not to depend on the clock time, then the time index of a decision can be a relative index that determines the target position that is associated with a decision rather than the clock time associated with the decision.

The above results apply to the case of an encounter between a target and a collection of detection systems. However, if the detection systems are not collocated, it is generally convenient to describe encounters of this kind in terms of encounters between the target and the individual detection systems. In either case, if the event target detection for a detection system is not independent of the event for other detection systems, then in order to describe this in an encounter model the correlation between the input to the detection system and the inputs to the other detection systems must be specified. This has been done in

some models as follows: First determine the probability of detection for each system acting alone. Let P_i be the probability that the ith system detects the target during the encounter under this condition. Next, consider two cases: In the first case, the random factors that determine detection for a system are independent of those that determine detection for the remaining systems. In the second case, the random factors that determine detection for the systems are completely dependent. In the first case, the probability that at least one system detects the target is given by: $P_I = 1 - (1 - P_1) \cdot (1 - P_2) \cdots (1 - P_n)$ where n is the number of detection systems involved. In the second case, the probability that none of the systems detect the target is given by: $1 - P_D = 1 - P_m$ where $P_m \geq P_i$ for $i = 1, 2, \cdots, n$ since if the mth system does not detect the target, none of the remaining systems ill detect it. The probability that at least one system detects the target is given by: $P = \alpha \cdot P_D + (1 - \alpha) \cdot P_I$ where α determines the degree of correlation and $0 \leq \alpha \leq 1$. A way to determine a value for α is described in Reference 8.

VII. Three Signal Excess Encounter Models

In the three models described in this section, detection is defined in terms of signal excess as it is in Section V. Each model determines a cumulative probability of detection for a target in an encounter with a passive sonar system. An observation in the models is indexed by time and the index can usually be considered to be the time at the end of the observation. During an encounter, observations are made of one or a series of localization regions. By implication, a false alarm can occur for a localization region that does not contain a target during an observation since the value of RD (or DT) is determined by some specified false alarm probability. However, as they are generally used, signal excess models do not account for false alarms. This can be viewed as equivalent to modeling the time to resolve a false alarm to be effectively zero.

To determine signal excess in the models, it is convenient to use Equation 2. For each decision in an encounter, there is a random variable X(t) defined by Equation 2 that determines the random character of the sig al excess. For a sequence of decisions, the set of these random variables ordered by their time index constitutes a stochastic process. And the joint distributions of these random variables determines the nature of the stochastic process. In the three encounter models described in this section, the stochastic process is called a lambda-sigma jump process. The time series that are generated by lambda-sigma jump processes are represented by the plot in Figure 2 below.

The jumps in the time series occur at times determined by a Poisson process with a mean jump rate lambda. This implies that the time between jumps is a random variable with an exponential distribution and that the expected times between jumps τ is equal to the reciprocal of lambda.
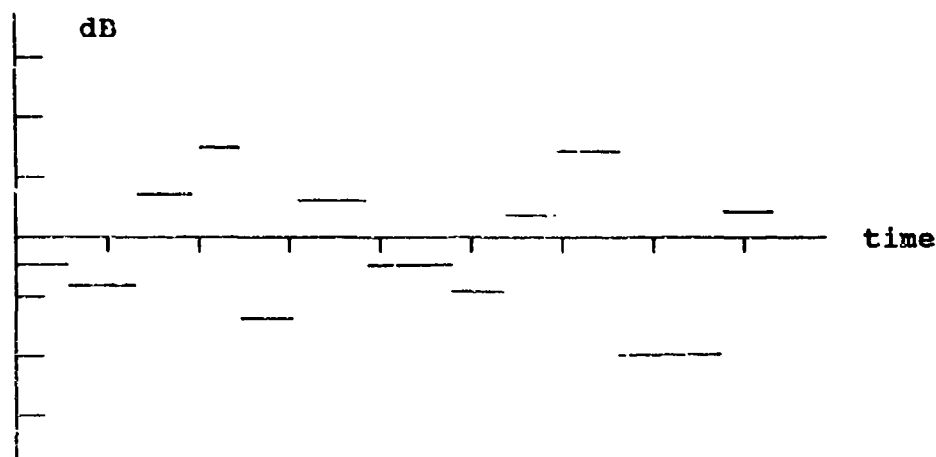

Figure 2. A time series representing a realization of a lambda-sigma jump process. On the plot, σ in dB equals one unit on the vertical axis and τ equals one time unit on the horizontal axis.

From Figure 2, note that the observed values of neighboring random variables are equal unless a jump has occurred between the observations. When a jump occurs, the first random variable after the jump is normally distributed with mean zero and variance σ^2 and it is independent of all the random variables before the jump. Conditioned on a jump pattern, this random variable and all the random variables between it and the next

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jump are dependent and the correlation coefficient between any pair is one. That is, if the value of the signal excess is known at some time, then all of the values between the last jump before that time and the first jump after that time are also known. However, since the jumps occur randomly, knowing the value of the signal excess with certainty at some time does not determine the values of the signal excess with certainty at neighboring times. In the unconditioned case, the correlation coefficient between the random variables $X(t)$ and $X(t+\tau)$ is equal to $1/e$. For this reason, τ is referred to as a relaxation time.

It appears that the use of the lambda-sigma jump process is based more on past practice than on experimental justification. In this regard, see Reference 9. By referring to Equation 1, it can be seen that the lambda-sigma jump process is determined by the sum of the stochastic processes that determine the random variables on the right side of this equation. Although the sum of a collection of normal random variables is a normal random variable, the sum of a collection of lambda-sigma jump processes is not a lambda-sigma jump process. This suggests that if the lambda-sigma jump process does adequately describe the variability of the signal excess, then the majority of the variability of the signal excess may be due to a single one of its elements. For example, transmission loss.

In the three encounter models described below, detection is defined in terms of signal excess as described in Section IV and decisions are indexed by a time that can usually be considered to

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be the time of the decision. During an encounter, observations
are made sequentially of one or a series of localization regions
(resolution cells). For a localization region that does not
contain a target, the signal observed during the observation of
the region is zero. For these observations, the time to resolve
a false alarm is zero. However, since the value of RD (or DT)
is finite and consequently the false alarm probability is not
zero, by implication, the cost associated with a false alarm is
not zero.

The First Passive Sonar Encounter Detection Model: This
model describes an encounter in terms of a series of decisions
with each decision based on the signal excess $X_{SE}(t)$ at a time
corresponding to the end of an observation. The observations are
of equal duration and the integration time that determines the
recognition differential is equal to the duration of the
observations. In the model, $X_{SE}(t)$ is determined by a lambda-
sigma jump process. For an encounter of m observations in
which SE(t) is unimodal and in which the time of the single
maximum is prior to or at the end of the encounter, it is shown
in Reference 10 that the probability p that detection will
occur during the encounter is given by the following equation:

(6) $p = 1 - [(1 - p_c)/(1 - \beta \cdot p_c)] \cdot (1 - \beta \cdot p_1) \cdots (1 - \beta \cdot p_m)$

where $\beta = 1 - \exp(-\delta t/\tau)$ and $p_i = \Phi[SE(t_i)/\sigma]$ for i = 1,2,
$\cdots$,m. Here, δt indicates the duration of an observation and
Φ indicates the standard normal cumulative distribution function
as before. The integer c is the index of a decision time t_c

for which $SE(t_c)$ is greater than or equal to $SE(t_i)$ for any time t_i and $t_1 \leq t_c \leq t_m$.

As τ approaches zero, β approaches one and Equation 6 approaches this form:

(7) $p = 1 - (1 - p_i) \cdots (1 - p_m)$.

In this limit, the signal excess random variables are all independent. Note that Equation 7 applies without the condition that $SE(t)$ be unimodal.

As τ approaches infinity, β approaches zero and Equation 6 approaches this form:

(8) $p = p_c$.

In this limit, the correlation coefficient between any pair of signal excess random variables is equal to one. Note that Equation 8 applies without the condition that $SE(t)$ be unimodal. Equation 8 defines a complete dependence encounter model.

The Second Passive Sonar Encounter Detection Model: This model is in a sense a third limiting form of the first passive sonar encounter detection model. In this limit, the time between decisions approaches zero. However, in this limit the integration time that determines the recognition differential is not equal to δt and it does not approach zero. It is, in effect, chosen by the user of the model through the user's choice of the value for the recognition differential. For an encounter that begins at t1 and ends at t2 and for which $X_{SE}(t)$ is determined by a lambda-sigma jump process and $SE(t)$ is

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unimodal, it is shown in Reference 10 that for this limit,
Equation 6 has the following form:

(9) $p = 1 - [1 - p(t_c)] \cdot \exp[-(1/\tau) \cdot \int_{t1}^{t2} p(t)\, dt]$

where $p(t) = \Phi[SE(t)/\sigma]$ and where now t_c is the encounter
time such that $SE(t_c)$ is greater than equal to $SE(t)$ for any
other encounter time t and $t1 \leq t_c \leq t2$.

The Third Passive Sonar Encounter Detection Model: This
model describes an encounter between a target and a passive sonar
detection system in which detection occurs during an encounter if
the average value of the square of the continuously observed
signal-to-noise ratio over a time interval of length u is
greater than or equal to the square of the signal-to-noise ratio
that determines the recognition differential for an integration
time equal to u. With $R(s)$ the random signal-to-noise ratio
at a time s and $R_m(u)$ the random signal-to-noise ratio that
determines the random recognition differential for an integration
time u, detection during an encounter occurs at the first time
t that the following inequality is satisfied:

(10) $(1/u) \int_{t-u}^{t} [R(s)/R_m(u)]^2\, ds \geq 1$

where the time origin is chosen so that $t \geq 0$ and where the
integration time $u = t$ for $t < t_0$ and $u = t_0$ for $t \geq t_0$
where t_0 is a maximum integration time. The random integrand
in the inequality is related to the random signal excess at the
time s for an integration time u. The relation is:

(11) $10 \log [R(s)/R_m(u)]^2 = 2[SE(s;u) + X(s)]$

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where $SE(s;u)$ is the expected value of the signal excess at a time s for an integration time u and $X(s)$ is the random component of the signal excess at the time s. In the model, $X(s)$ is determined by a lambda-sigma jump process and $SE(s;u)$ is determined by an expected value sonar equation with a recognition differential $RD(u) = 10 \log r_m(u)$. Here, $r_m(t)$ is the value of the signal-to-noise ratio that gives a probability of detection equal to .5 for an integration time t and a specified probability of false alarm p_f. With the signal detection process described by a Case II signal detection model, the detection index necessary to give the required operating point $(p_f, .5)$ is related to the integration time t and the signal-to-noise ratio $r_m(t)$ by:

(12) $\quad d = u \cdot (BW) [r_m(t)]^2$

where BW is the bandwidth of the receiver. For a spectrum analyzer, BW would be the bandwidth corresponding to a given frequency resolution and d would be the detection index required in order to be at the operating point $(p_f, .5)$ for a signal that was contained within a bandwidth BW. Since d in Equation 12 must be the same for $t = u$ and $t = t_0$,

(13) $\quad RD(u) = 5 \log(t_0/u) + RD(t_0)$

where t_0 is the maximum integration time. Then, since $SE(s;u) - SE(s;t_0) = RD(t_0) - RD(u)$, by using Equation 13 and Equation 11, Relation 10 becomes:

(14) $\quad \int_{t-u}^{t} 10^{(1/5)[X(s) + SE(s;t_0) - 5 \log(t_0)]} \, ds \geq 1$

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where as above the time origin is chosen so that $t \geq 0$, the integration time $u = t$ for $t < t_0$ and $u = t_0$ for $t \geq t_0$ and where $SE(s;t_0)$ is the expected value of the signal excess at the time s for a recognition differential determined by an integration time t_0. In an encounter, detection occurs the first time that Relation 14 is satisfied.

As is pointed out in Reference 11, the appeal of the Third Passive Sonar Encounter Detection Model relative to the Second and First Passive Sonar Encounter Detection Models is that it appears to more closely describe the detection process in passive sonar detection systems that display their processed data to an operator in a continuous manner over a time window of duration t_0. However, results reported in Reference 12 indicate that the difference between the three models may not be significant in some types of encounters.

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VIII. Straight Line Encounters

Suppose a target's detectability depends on its range from a detection system and that the probability of detection is effectively zero beyond a range r_m for any target azimuth. In this report, an encounter between the target and the detection system is the event that the range between the target and the detection system is less than or equal to r_m. In addition, suppose r_m is small enough so that when the target and the detection system are having an encounter they can be considered to be moving on planes parallel to a tangent plane to the earth's surface at some point in their vicinity. If this is the case, then while the target and detection system maintain a constant course and speed during an encounter, the encounter is called a straight line encounter.

A straight line encounter can be described in terms of a two dimensional rectangular coordinate system whose plane is parallel to the tangent plane to the earth. If the coordinate system is stationary relative to the detection system with the detection system located at the origin and is oriented so that the target's motion is parallel to the y-axis and is in the positive y-direction, then the target's x-coordinate during a straight line encounter will be constant. The constant is equal to the target's horizontal range at the closest point of approach (CPA) on the straight line track on which the target is moving relative to the detection system during the encounter. This range is called the target's lateral range.

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A complete straight line encounter is a straight line encounter that begins at a range from a detection system that is greater than or equal to r_m and continues past CPA to a range from the detection system that is again equal to or greater than r_m. Let $p(x)$ be the cumulative probability that a target is detected by a detection system in a complete straight line encounter in which the target's lateral range is x. Then the function $p(x)$ defines what is called a **lateral range curve** or **lateral range function**.

Let p be the probability that a target is detected during a complete straight line encounter. If the lateral range of a target in a straight line encounter is assumed to be a continuous random variable X with a uniform distribution with $f_X(x) = 1/a$ for $|x| \leq a/2$ and $p(x) = 0$ for $|x| > a/2$, then the probability that a target will be detected during a complete straight line encounter is given by:

(15)
$$p = (1/a) \int_{-\infty}^{\infty} p(x)\ dx$$

where the limits of integration can be used since the value of $p(x)$ is zero for $|x| > a/2$. Equation 15 suggests a measure of a detection system's capability to detect a target in a straight line encounter. The measure W is called **sweep width** and

(16)
$$W = \int_{-\infty}^{\infty} p(x)\ dx.$$

With this definition, Equation 16 becomes: $p = (1/a) \cdot W$.

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IX. Two Intermittent Signal Encounter Models

In the intermittent signal encounter models that are described in this section, an encounter is a complete straight line encounter, and during an encounter a target either emits a signal (an acoustic transient) or its presence (a visible submarine mast) is the cause cf a signal at various times. Two cases are considered: In the first case, the signals occur periodically, the signals are of length δt and the time between the occurrence of signals is τ where $\tau > \delta t$. In the second case, $\delta t = 0$ (the signals are instantaneous) and the signals occur at times determined by a Poisson process for which the expected time between signals is equal to τ. In the model, the detectability of a target signal depends on a target's horizontal range from a detection system, but on no other factors. If a signal occurs while the target is within a range r, it will be detected. For a continuous signal the lateral range function of a detection system for a target is: $p(x) = 1$ for $|x| \leq r$ and $p(x) = 0$ for $|x| > r$ where the horizontal range r is determined by the characteristics of the detection system and the target. The geometry for an encounter is shown in Figure 3 below.

For intermittent signals, the length of a target's track relative to a detection system on which a signal will be detected is $2 \cdot (r^2 - x^2)^{\frac{1}{2}} + w \cdot \delta t$ where w is the speed of the target relative to the detection system. So, a target's exposure time during an encounter is $(2/w) \cdot (r^2 - x^2)^{\frac{1}{2}} + \delta t$.

For periodic signals, there are two cases. In the first
case, $r \geq w \cdot (\tau - \delta t)/2$. In this case, the signals result in the
following lateral range function:

$$p(x) = 0 \quad \text{for} \quad |x| > r$$

(17) $\qquad p(x) = 1 \quad \text{for} \quad |x| < (r' - [w \cdot (\tau - \delta t)/2]')^{\frac{1}{2}}$

$$p(x) = [2/(w \cdot \tau)] \cdot (r' - x')^{\frac{1}{2}} + \delta t/\tau \quad \text{otherwise}$$

In the second case, $r < w \cdot (\tau - \delta t)/2$ and the middle equality
in Equation 17 does not apply.

For signals that are instantaneous and whose occurrence is
determined by a Poisson process, the signals result in the
following lateral range function:

$$p(x) = 1 - \exp(-[2/(w \cdot \tau)] \cdot (r' - x')^{\frac{1}{2}}) \quad \text{for} \quad |x| \leq r$$

(18) $\qquad p(x) = 0 \quad \text{for} \quad |x| > r.$
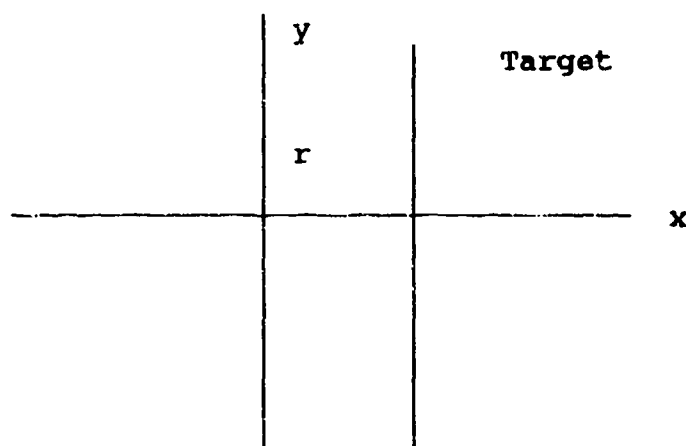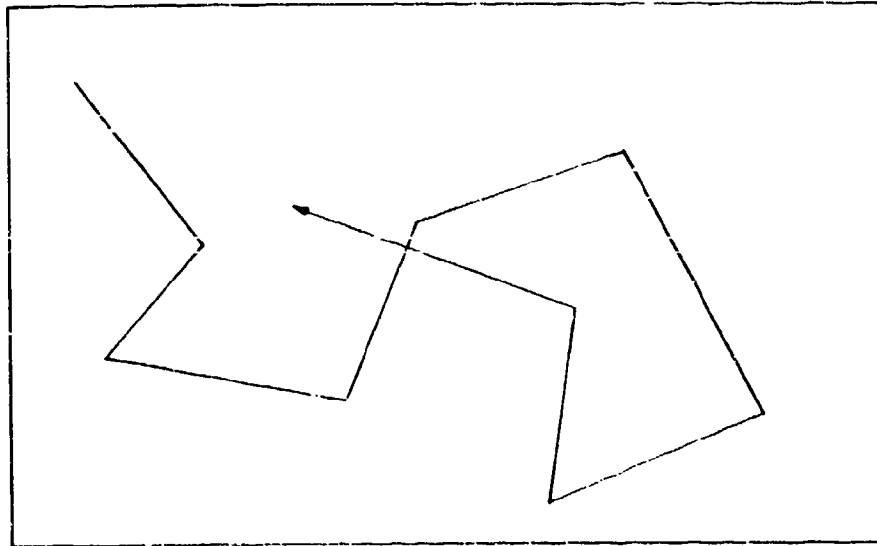

y

Target

r

x

Figure 3. The encounter geometry for the two intermittent signal
models described here.

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For signals whose occurrence is determined by a Poisson process and for which $\delta t > 0$, signals can overlap. If this is allowed, then Equation 18 can be modified to describe this case by adding $\delta t/\tau$ to the term in the exponent of Equation 18 that is within the square brackets. In particular, note that this modified Equation 18 can be approximated by the bottom equality in Equation 17 when $(2/w \cdot \tau)(r^2 - x^2)^{\frac{1}{2}} + \delta t/\tau \ll 1$. This implies that when the expected time τ between signals is large relative to the exposure time $(2/w)(r^2 - x^2)^{\frac{1}{2}} + \delta t$, the periodic signal model and the Poisson random signal model are effectively equivalent.

If $\delta t = 0$ and $r < (w \cdot \tau)/2$, for the periodic intermittent signal, $W = \pi \cdot r^2/(v \cdot \tau)$. With the above approximation, this is also the sweep width for the Poisson random signal model.

X. A Random Search Model

A search of a region in which a target moves on a track that consists of a number of straight segments placed in such a way that in a limiting sense every section of the region is equally likely to be searched on a segment is referred to as a random search in Reference 6. Representation of a search region with the track segments that could be imagined to be the tracks of a random search are shown in Figure 4.

Figure 4. A search region and a track that could be described as a random search track.

Two developments of a model to describe this kind of search are contained in this section. The first development is based on the following conditions: 1. A target is at a fixed position within a defined search region. 2. A searcher's track is a

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sequence of straight line segments that are within the search
region.   3. The searcher's detection system is such that while on
a track segment, a rectangle is searched that is contained within
the search region, is of length equal to the length of the track
segment and is oriented so that its long axis is parallel to the
track segment.   4. The probability that the searcher's detection
system will detect a target while on a track segment with a
search rectangle that does not contain the target is zero.   The
probability that the searcher's detection system will detect a
target while on a track segment with a search rectangle that
contains the target is p(x) where x is the target's lateral
range for the track segment and p(x) is the lateral range curve
for a complete straight line encounter lateral range x.   A
representation of a search rectangle is shown in Figure 5 below.
5. The track segments are located in such a way that the event
that the target is within the search rectangle associated with a
track segment is independent of the event that the target in the
search rectangle associated with any other track segment.   And
the probability of the event is equal to the ratio of the area of
the search rectangle to the area of the search region and, given
a target is within a search rectangle, its position is uniformly
distributed over the rectangle.

Condition 4 implies that the random search model is based
on the concept of a complete straight line encounter.   The
definition of an encounter that is intended here is that given in
Section VI.   This implies that in the random search model the

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time to resolve a false alarm is zero. However, for the model, p_d and p_f are considered to be determined by some criterion such that p_f is less than one. Consequently, although the time to resolve a false alarm is zero in the model, the cost associated with a false alarm is not zero. (A simple model that accounts for the time to resolve false alarms is described in Reference 13.) Condition 4 also implies that when a searcher is on a track segment with a search rectangle that contains a target, the encounter is a complete straight line encounter. And Condition 5, which can be considered to specify a random arrangement of the track segments, implies that when this is the case, for the complete straight line encounter, the target's lateral range is a random variable that is uniformly distributed between $-b/2$ and $b/2$ where b is width of the search rectangle (the dimension of the rectangle perpendicular to the associated track segment).

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┌─────────────────────────────────────┐
│                                      │
│             Swept Area               │
│                                      │
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                  Track
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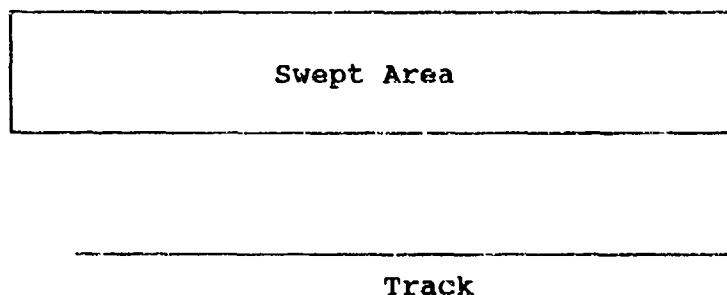
Figure 5. A track segment and its associated search rectangle. that could correspond to a search with an aircraft mounted infrared detection system.

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Based on the above considerations, the probability that a target will be detected while a searcher is on a track segment with an associated search rectangle that contains the target is given by:

(19) $$\int_{-\infty}^{\infty} p(x)\ f_X(x)\ dx = W/b$$

where $f_X(x) = 1/b$ for $-b/2 \leq x \leq b/2$ and $f_X(x) = 0$ and $p(x) = 0$ otherwise. Note that the left side of Equation 19 applies to any complete straight line encounter in which the target's lateral range for the encounter is considered to be a random variable with a distribution determined by the probability density function $f_X(x)$. If it is not given that the target is within the search rectangle associated with a track segment, then the unconditional probability that the target will be detected on the track segment is given by: $(W/b) \cdot (\delta A/A)$ where δA is the area of the search rectangle associated with the track segment and A is the area of the search region. With l the length of the rectangle, $\delta A = b \cdot l$ and the probability becomes: $(W \cdot l)/A$. Then, since the event that the target will be in the search rectangle of a track segment is independent of the event that it will be in the search rectangle of any other track segment, the probability p that a random search consisting of m track segments will detect the target is given by:

$1 - [1 - (W \cdot l_1)/A][1 - (W \cdot l_2)/A] \cdots [1 - (W \cdot l_n)/A]$ where l_i is the length of the ith track segment. The probability is also given by: $p = 1 - \exp(\Sigma\ \ln[1 - (W \cdot l_i)/A])$ where the sum

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index $i = 1, 2, \cdots, n$. If $(W \cdot l_i)/A \ll 1$ for $i = 1, 2, \cdots, n$, then this expression can be approximated by:

(20) $p = 1 - \exp[-(W \cdot l)/A]$

where $l = \Sigma \; l_i$ is the track length of the search. Equation 20 is known as the random search formula.

The second development of the random search formula is based on Equation 5 and a detection rate for a random search given by: $r(t) = W \cdot v(t)/A$. With this detection rate and Equation 5, the random search formula is given by:

(20a) $P(T \leq t) = 1 - \exp \{-[W \cdot l(t)]/A\}$

where $l(t)$ is the track length for a random search that starts at time 0 and ends at time t and

(20b) $l(t) = \int_0^t v(s) \; ds.$

Replacing $P(T \leq t)$ by p and $l(t)$ by l gives Equation 20. In the form of Equation 20a, the random search formula indicates explicitly the relation between the probability of detection and the duration of a random search. Note that Equation 20a implies that the sweep width is independent of speed over the range of speeds in the encounter.

As an example application, consider the periodic signal model of Section IX with $\delta t = 0$, $r < (w \cdot \tau)/2$ and $v(t) = v$. For this case, $p = 1 - \exp[-(\tau \cdot r'/A) \cdot (t/\tau)]$.

Reference 14 contains an example of an application of the method used in the second development of the random search formula to a random search where the search region expands with time.

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XI. Ladder and Barrier Search Models

In some barrier searches, the barrier search track is a ladder search track relative to a reference system that moves with the target. This fact is used in the barrier search model development that follows the two ladder search model developments below. The first ladder search model is referred to as an ideal ladder search model because of the idealizations that are involved in its description of a ladder search. The second ladder search model is referred to as a degraded ladder search. It can be considered to describe a ladder search track in which navigational errors result in omissions and overlaps in coverage.

An Ideal Ladder Search Model: The model is based on the following conditions: 1. A ladder search region is a rectangle that contains a fixed target. 2. During a search of the region, the searcher's detection system moves on a set of m parallel track segments of length b separated by a distance s. 3. As the detection system moves along a track segment, it searches a rectangular strip of length b and width s within the search region. 4. The m rectangular strips that correspond to the m track segments completely cover the ladder search region with no overlap. 5. If a target is within the rectangular strip corresponding to a track segment, then there will be a complete straight line encounter between the target and the detection system when the detection system moves along the track segment and the lateral range of the encounter will be uniformly distributed across the width of the strip. If the target is not

in the rectangular strip, then there will not be an encounter and
the probability that the target will be detected while the
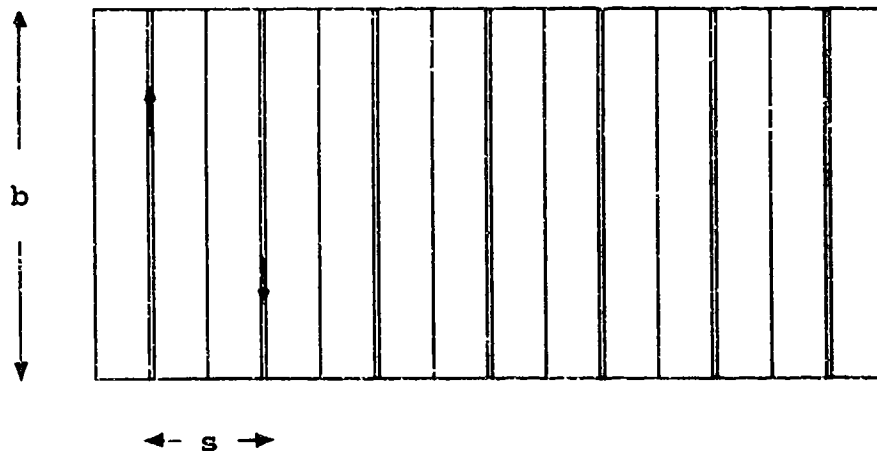detection system is on the track segme it is zero.

Figure 6.   A schematic representation of a ladder search geometry
for a case in which the ladder search track segments are
superimposed on and bisect their corresponding rectangular
strips.

Since targets ottside of the rectangular strip that
corrasponds to a track segment cannot be detected while a
detection system is on the track segment because of Condition 5,
in the model, the sweep width W of a searcher's detection
system must satisfy the relation $W \leq s$. In particular, W = s
only holds when the detection system detects a target that is in
a rectangular strip with probability one for ary target lateral
range.   This kind of detection system is sometimes referred to as
a cookie cutter detection system.   However, this terminology can

be misleading since it suggests the detection system detects equally well for all azimuths. But this is not a requirement on the system in order that $W = s$.

The ideal ladder search model implies that if the conditions of the model are satisfied, then the probability p that a target will be detected by a an ideal ladder search is given by:

(21) $p = W/s$

where $W/s \leq 1$. The quantity W/s is called the coverage factor in this case.

A Degraded Ladder Search Model: The above model implies perfect navigation in addition to other idealizations. A model of a ladder search is given in Reference 6 that could be used for cases in which this is a poor assumption. The model which is referred to here as a degraded ladder search model can be considered to describe navigational inaccuracies in terms of omissions and overlaps of the rectangular strips. It can be developed as follows: Consider a random search in the ladder search region whose track length is equal to the search track length required to complete an ideal ladder search, that is, a track length $l = m \cdot b$. The degraded ladder search model describes the result of omissions and overlaps in a ladder search to be such that the probability of detection for this random search is equal to the probability of detection for the degraded ladder search. Consequently, since the area of the ladder search region is $m \cdot s \cdot b$, for the degraded ladder search model:

(22) $p = 1 - \exp(-W/s)$.

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Here, the requirement that the coverage facto $W/s \leq 1$ for Equation 21 can be relaxed. However, it should still be considered as an approximate condition.

The condition that the target be fixed within the rectangular search region is critical to both Equation 21 and Equation 22. However, these results are also applicable to a search for a moving target under the conditions that are described next.

A Barrier Search Model: A target moves with a constant course and a constant speed u. Both the target's course and the target's speed are known by a searcher. The searcher establishes a barrier of width b that is perpendicular to the target's track and moves on the barrier with a speed $v > u$. The barrier is designed so that in a reference system relative to the target the barrier search is a ladder search that satisfies the conditions for a ladder search that are given above. There are two cases to consider: 1. The barrier is established in front of the target. 2. The barrier is established behind the target.

From the search geometry for a barrier established in front o. the target, it can be seen from Figure 7 below that $\theta = \sin^{-1}(u/v)$ and $d = v \cdot \tau$ where $\tau = s/(v + u)$ is the time to move from one search leg to the next. The angle θ and the perpendicular distance d that depend on u, v and s, and the width of the barrier b are the quantities that are required in order to establish the barrier operationally.
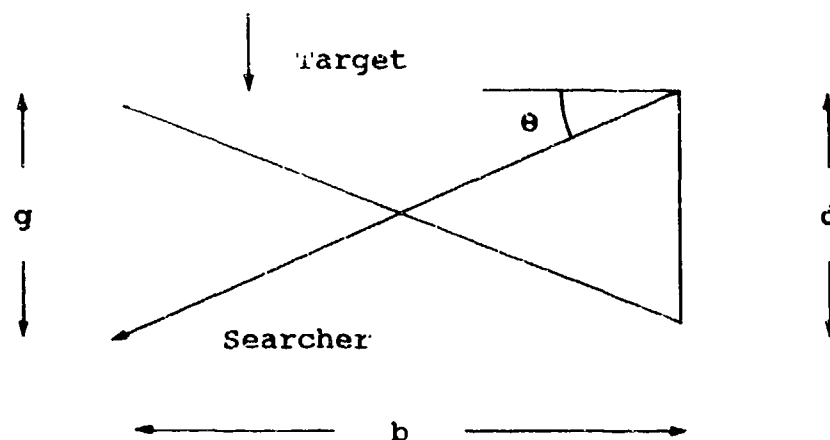
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Figure 7. A barrier search track shown for a barrie established in front of the target. The track is shown in a reference system fixed relative to the earth.

For a barrier that is established in front of a target, one of three barrier types will result. A barrier's type is determined by the relation of the distance d to the distance $g = ut$ where the time $t = b/(v^2 - u^2)^{\frac{1}{2}}$ is the time to complete a search leg (cross the barrier). The barrier type is determined as follows: 1. For $g < d$, the barrier is an advancing barrier. 2. For $g = d$, the barrier is a stationary barrier. 3. For $g > d$, the barrier is a retreating barrier.

For a barrier established behind the target, there is only one barrier type and it is called an overtaking barrier. For an overtaking barrier, $\theta = \sin^{-1}(u/v)$ as for a barrier established in front of the target. But, for an overtaking barrier, $t = s/(v - u)$ and $d = v \cdot s/(v - u)$.

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Given the target crosses the barrier, the probability of detection for an ideal barrier search is given by Equation 21 and the probability for a degraded barrier search is given by Equation 22 where the terminology refers to the nature of the ladder search in the reference system moving with the target. A discussion of an application of these two equations to a search for a magnetic anomaly target is given in Reference 15.

XII. A Target State Estimation Procedure

A target state estimation procedure based on bearing observations is developed in this section that generates point estimates of a target's position and velocity vector coordinates in a rectangular coordinate system. The procedure is based on a model in which bearing errors are unknown and are not determined by random variables with known distributions. Because of this, confidence regions for the estimates are not generated by the procedure. However, for a moving target, it illustrates general characteristics of bearings only target motion analysis (TMA). The model is defined as follows: 1. The target moves in a plane with a constant but unknown course and speed. 2. Observations of the target are made from known positions at known times. 3. The observations provide only target bearings with unknown errors. The model geometry is shown in Figure 8.
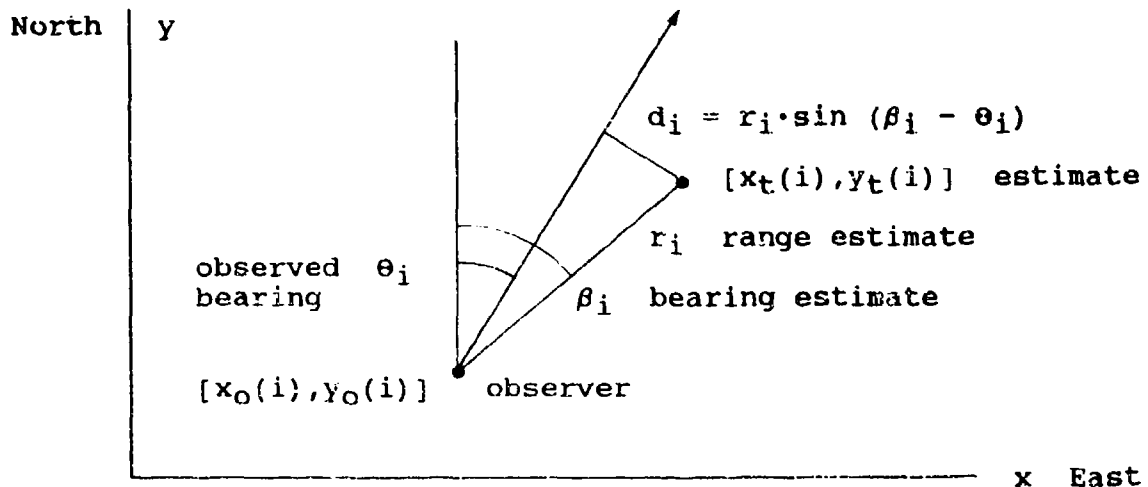

Figure 8. The geometry of the target motion analysis model.

The procedure criterion is: For observations from n positions, choose target position estimates and target velocity component estimates u_x and u_y that make the sum of the squares of the algebraic distance between the estimated positions and their corresponding observed bearing lines a minimum. From Figure 8, it can be seen that the algebraic distance can be written as $d_i = [x_t(i) - x_o(i)] \cdot \cos \theta_i - [y_t(i) - y_o(i)] \cdot \sin \theta_i$. Because of the requirement that the target move with constant course and speed during the encounter, the number of independent estimates is reduced from $2n$ to 4, u_x, u_y and any two position estimates $x_t(j)$, $y_t(j)$. In the following development, $j = 1$ and with $i = 2,3, \cdots ,n$ the remaining estimates are given by: $x_t(i) = x_t(1) + u_x \cdot (t_i - t_1)$ and $y_t(i) = y_t(1) + u_y \cdot (t_i - t_1)$. To determine "best" estimates of the target state parameters, take the partial derivative of the sum $S = \Sigma (d_i)^2$ with respect to each of them. Then set the four partial derivatives equal to zero. This creates four linear equations in $x_t(1)$, $y_t(1)$, u_x and u_y whose solution are the desired estimates $x_t(1)$, $y_t(1)$, u_x and u_y. In matrix notation, the equations can be represented by $AX = B$ where the elements of X are: $x_{11} = x_t(1)$, $x_{21} = y_t(1)$, $x_{31} = u_x$ and $x_{41} = u_y$. A necessary condition for a unique solution for X is that $n \geq 4$. Otherwise, the determinant of A will be equal to zero. The procedure can also be used if a target's course and speed are constant and known and, in particular, if the target is stationary so that u_x and u_y both equal zero. In this case,

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since the number of unknowns is two, the number of linear equations is also two and a necessary condition for a unique solution is $n \geq 2$.

Now, suppose the observations are at positions and times that correspond to the positions and times of an observer moving on some constant course at some constant speed (including zero speed). In this case, the observation position coordinates are related by the following equations: $x_o(i) = x_o(1) + v_x(t_i - t_1)$ and $y_o(i) = y_o(1) + v_y(t_i - t_1)$ where v_x and v_y are the required velocity components of the observer. Using these equations of motion, the matrix equation $AX = B$ can be transformed to the matrix equation $AX' = 0$ where the elements of the matrix X are related to the elements of the matrix X' by the equations: $x'_{11} = x_t(1) - x_o(1)$, $x'_{21} = y_t(1) - y_o(1)$, $x'_{31} = u_x - v_x$ and $x'_{41} = u_y - v_y$.

Since the linear equations represented by $AX' = 0$ are homogenous, they do not have unique solutions and consequently neither do the equations represented by $AX = B$. However, if there is at least one observation whose time and position is not determined by the above equations of motion, then the transformation from X to X' cannot be made, and in general a unique solution for X can be found. If the observations are made from a platform that is moving with a constant course and speed, this condition can be achieved by either changing the course, the speed or both prior to completing the observations.

Estimation models that describe bearing error as a random

variable provide a basis for determining confidence regions for point estimates. A model is developed in Reference 16 that does this for either target bearing observations made from two or more points simultaneously or for a target that is stationary relative to the observation points.

XIII. Position Distributions That Change with Motion

Target motion models provide a basis for determining position distributions that change with target motion. In this section, two classes of target motion models are considered. In the first class, a target moves in a plane with a constant course and speed and the course and speed are independent of the target's position. In the second class, a target moves in a plane but its course or speed changes during the motion. Three members of the first class are developed first. This is followed by a brief discussion of some models of the second class.

Motion Models of the First Class: For the first class of motion models, the joint density function of the distribution that determines a target's coordinates $X(t)$ and $Y(t)$ at some time $t \geq 0$ can be determined by:

(23) $f_{X(t),Y(t)}(x,y;t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X(0),Y(0)}(q,s;0) \, f_{V,W}(v,w) \, dvdw$

where $V = U_x$, $W = U_y$ and U_x and U_y are the random variables that determine the target's velocity components u_x and u_y and $v = u_x$, $w = u_y$, $q = x - v \cdot t$ and $s = y - w \cdot t$. Equation 23 can be developed as follows: To first order, $f_{X(t),Y(t)}(x,y;t) \, \delta x \delta y$ is the probability that a target's coordinates are in an element of area $\delta x \delta y$ and for given values of v, w and t, target positions in an element of area $\delta q \delta s$ will be translated to an element of area $\delta x \delta y$ that is identical in form and size to $\delta q \cdot \delta s$. And so, to first order, $f_{X(0),Y(0)}(q,s;0) \, \delta x \delta \cdot \, f_{V,W}(u,v) \, \delta v \delta w$ is the probability that the target's coordinates at time 0 are in an element of area

δqδs that is located such that the target's coordinates will be in the element of area δxδy at time t since x = q + v·t and y = s + w·t. And, to first order, the sum of such probabilities for all pairs of values of v and w is also the probability that the target's coordinates at time t are in the element of area δxδy. In the limit after equating the two expressions for this probability and cancelling the common factor δx·δy, Equation 23 results.

The First Motion Model: In the first model, X(0) and Y(0) are both independent normal random variables with means μ_x and μ_y and equal standard deviations σ. However, U_x and U_y are not normal and they are not independent random variables. In this model, U_x = u·sin φ and U_y = u·cos φ where φ is the random variable that determines the target's course and u is the targets speed which is known. So, only a value for the random variable φ is required to determine the target's velocity. In the model, φ has a uniform distribution over the interval 0 to 2π and it is convenient to chose the rectangular coordinate system so that the means μ_x and μ_y are each equal to 0. Then, with the circular normal distribution determining the random position coordinates and with the distribution that is described above determining the random velocity components , in the coordinates u and φ, the integral of Equation 23 is a single integral over φ and the integrand of the integral is $(1/2\pi\sigma')$ exp$[-(q' + s')/2\sigma']$ $(1/2\pi)$ where

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now $q = x - u \cdot t \cdot \sin \phi$ and $s = y - u \cdot t \cdot \cos \phi$. Integration gives $f_{X(t),Y(t)}(x,y;t)$ as:

(24) $(1/2\pi\sigma^2) \exp(-[x^2 + y^2 + (u \cdot t)^2]/2\sigma^2) I_0[(x^2 + y^2)^{\frac{1}{2}} \cdot u \cdot t/\sigma^2]$

where $t \geq 0$ and I_0 indicates the hyperbolic Bessel function of zeroth order. In Reference 6, $f_{X(t),Y(t)}(x,y;t)$ is plotted for several values of t in terms of $r = (x^2 + y^2)^{\frac{1}{2}}$, the target's range from the origin. The plots show a characteristic of the distribution that can be indicated as follows: First, replace $(x^2 + y^2)^{\frac{1}{2}}$ by r in $f_{X(t),Y(t)}(x,y;t)$. Next, multiply and then divide $f_{X(t),Y(t)}(r;t)$ by $\exp(-r \cdot u \cdot t/\sigma^2)$. This gives:

(25) $1/(2\pi\sigma^2)\exp(-[1/(2\sigma^2)](r - u \cdot t)^2)I_0(r \cdot u \cdot t/\sigma^2)\exp(-r \cdot u \cdot t/\sigma^2)$

where $t \geq 0$. As noted in Reference 17, $I_0(z) \cdot \exp(-z)$ is a slowly decreasing function that asymptotically approaches $1/(2\pi z)^{\frac{1}{2}}$ as z increases. The consequence of this is that a plot of $f_{X(t),Y(t)}(r;t)$ against r for values of t greater than $4 \cdot \sigma/u$ has the appearance of a normal density function.

A target's random rectangular coordinates $X(t)$ and $Y(t)$ and its random bearing $\theta(t)$ and range $R(t)$ from the origin are related by: $X(t) = R(t) \cdot \sin \theta(t)$ and $Y(t) = R(t) \cdot \cos \theta(t)$. Using these relations, $f_{X(t),Y(t)}(x,y;t)$ can be transformed to the joint density function $f_{R(t),\theta(t)}(r,a;t)$ of the random variables $R(t)$ and $\theta(t)$. To do this, replace $x^2 + y^2$ by r^2 in Expression 24. Then multiply the resulting expression by r, the Jacobian of the transformation. This gives the expression:

(26) $(1/2\pi)(r/2\pi\sigma^2) \exp(-[r^2 + (u \cdot t)^2]/2\sigma^2) I_0(r \cdot u \cdot t/\sigma^2)$

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where $0 \le r$ and $0 \le \alpha \le 2\pi$. And, this is the joint density function $f_{R(t), \Theta(t)}(r, \alpha; t)$. By inspection, the marginal density function of $f_{\Theta(t)}(\alpha; t)$ of $\Theta(t)$ is $1/2\pi$ over the interval 0 to 2π and the marginal density function $f_{R(t)}(r; t)$ of $R(t)$ is Expression 26 multiplied by 2π.

The Second Motion Model: In the second model, $X(0)$ and $Y(0)$ are independent normal random variables with means μ_x and μ_y and standard deviations σ_x and σ_y that determine a target's random position coordinates at time 0. And U_x and U_y are independent normal random variables with means $\hat{u}_x$ and $\hat{u}_y$ and standard deviation σ_u that determine a target's random velocity components. Because of these conditions, the target coordinates are $X(t) = X(0) + U_x \cdot t$ and $Y(t) = Y(0) + U_y \cdot t$ at time t. This implies that $X(t)$ and $Y(t)$ are independent normal random variables with means $\mu_x + \hat{u}_x \cdot t$ and $\mu_y + \hat{u}_y \cdot t$ and with standard deviations $\sigma_x^2 + \sigma_u^2 \cdot t^2$ and $\sigma_y^2 + \sigma_u^2 \cdot t^2$. The model describes a bivariate normal position distribution whose center moves with a constant velocity determined by $\hat{u}_x$ and $\hat{u}_y$ and which becomes more and more circular as its standard deviations increase with the passage of time. Although the targets joint density can be found by using Equation 23, this procedure is more direct. For another discussion of the first and second models, see Reference 7.

The Third Motion Model: In the third model, the target is at the origin of a rectangular coordinate system at time zero. After that, its position is uniformly distributed on a circular

disk of radius $u_m \cdot t$ centered at the origin. This implies that the joint density function of the position distribution is:

$$(27) \qquad f_{X(t),Y(t)}(x,y;t) = 1/(\pi \cdot u_m^2 \cdot t^2)$$

for $t > 0$ where $x^2 + y^2 \le u_m^2 \cdot t^2$ and that the joint density function of the distribution of the random variables $\Theta(t)$ and $R(t)$ that determine a target's bearing from the origin is:

$$(28) \qquad f_{R(t),\Theta(t)}(r,a;t) = r/(\pi \cdot u_m^2 \cdot t^2)$$

for $t > 0$ where $0 < r \le u_m \cdot t$ and $0 \le a < 2 \cdot \pi$. Since the values of r and a are independent and the joint density function is equal to the product of $1/(2 \cdot \pi)$ and $2 \cdot r/(u_m^2 \cdot t^2)$, the random variables $R(t)$ and $\Theta(t)$ are independent and $f_{R(t)}(u) = 2 \cdot r/(u_m^2 \cdot t^2)$ where $0 < r \le u_m \cdot t$ and $f_{\Theta(t)}(a) = 1/(2 \cdot \pi)$ where $0 \le a < 2 \cdot \pi$. These two marginal distributions can be achieved by choosing at time 0 a course ϕ from the uniform distribution with density: $f_\phi(\phi) = 1/(2 \cdot \pi)$ where $0 \le \phi < 2 \cdot \pi$ and a speed U from the triangular distribution with density: $f_U(u) = 2 \cdot u/u_m^2$ where $0 \le u \le u_m$. These choices defin the third motion model.

Motion Models of the Second Class: For the second class of motion models, a target's course or speed or both can change. In general, a monte carlo simulation method is required in order to determine a position distribution that is based on such a model. As an example of cases in which the distribution can be described analytically, see Reference 18 and Reference 19. As an example of a case in which it can not, suppose a target's initial position is described in terms of a number assigned to a

subregion in the xy-plane where the number assigned represents the probability that the subregion contains the target at an initial time. In addition, suppose for each subregion a course and speed distribution is determined by assigning numbers to course and speed pairs where a number represents the probability the target will have the course and speed at the initial time given it is in the subregion at that time. Next suppose for each course and speed pair there is a time distribution that determines the duration of the course and spe d pair and that the time distribution is determined by a number assigned to each discrete time point where the number represents the probability that the target's course and speed pair will be determined by a new course and speed distribution. By extending this kind of procedure and then implementing it in a monte carlo simulation, one can generate complex position distributions that describe a target's position at discrete time points.

XIV. Position Distributions That Change with Search

For the models that are considered here, a target is within a region that has been divided into subregions or cells. And, for each cell, a number has been assigned to the cell that is interpreted as the probability that the target is within the cell. The set of these probabilities constitute a position distribution. Suppose information becomes available that a search has been conducted for the target and that the target has not been detected. Or suppose the information is that the target has been detected. In the first case, negative information is available that can be used to modify the position distribution. In the second case, positive information is available that can be used to modify the position distribution.

Position Distributions and Negative Information: For a region that contains a target and consists of n subregions, let the event S_i = (the target is in the ith subregion). And let the event $\bar{C}$ = (no contact). Then, given no contact in a search of the region, the targets's position distribution can be modified as follows:

$$(29) \quad P(S_i|\bar{C}) = P(\bar{C}|S_i) \cdot P(S_i)/P(\bar{C})$$

where $i = 1,2, \cdots ,n$ and $P(\bar{C}) = \Sigma\ P(\bar{C}|S_j) \cdot P(S_j)$ with the sum index $j = 1,2, \cdots ,n$. Note that Equation 29 can be obtained by using Bayes theorem. To illustrate how Equation 26 might be used, suppose that a search in a subregion is considered to be a random search and that the sweep width of detection system against the target depends on subregion being searched. For this

case, let A_i be the area of the ith subregion and let W_i be the sweep width in that subregion. Then, given no contact in a search of a subregion, $P(\bar{C}|S_i) = \exp[-(W_i \cdot l_i)/A_i]$ where l_i is the track length of the searcher in the ith subregion. Given values for $P(S_i)$, W_i/A_i and l_i for $i = 1, 2, \cdots, n$, a position distribution can be determined that has been modified by the negative information.

Position Distributions and Positive Information: In the case of positive information, the event $C = \{a\ contact\}$ occurs. Then, given a contact in a search of the region, the target's position distribution can be modified as follows:

(30)　$P(S_i|C) = P(C|S_i) \cdot P(S_i)/P(C)$

where $P(C) = \Sigma\ P(C|S_j) \cdot P(S_j)$ and the sum index $j = 1, 2, \cdots, n$. To illustrate a way Equation 30 might be used, suppose that a contact is a line of bearing detection or an omnidirectional sensor detection and that the cells are range cells. Then, with r_i a range identifying the ith cell and R a random variable, $P(S_i|C)$ becomes $P(R = r_i|C)$, $P(C|S_i)$ becomes $P(C|R = r_i)$ and $P(S_i)$ becomes $P(R = r_i)$. In a continuous analog of this, $f_R(r|C)$ replaces $P(S_i|C)$, $P(C|R = r)$ replaces $P(C|S_i)$ and $f_R(r)$ replaces $P(S_i)$.

Position Distributions and Uncertain Information: The event C is the union of the two events: $T_C = \{a\ true\ contact\}$ and $F_C = \{a\ false\ contact\}$. And the event $\bar{C}$ is the union of two events: $T_{\bar{C}} = \{a\ true\ no\ contact\}$ and $F_{\bar{C}} = \{a\ false\ no\ contact\}$ The correspondence between C, $\bar{C}$, S_i, $\bar{S}_i$, T_C, F_C, $T_{\bar{C}}$ and

$F_{\bar{C}}$ and the events of the Venn diagram in Figure 1 are as follows: C corresponds to D_1, $\bar{C}$ corresponds to D_0, S_i corresponds to H_1, $\bar{S}_i$ corresponds to H_0, T_C corresponds to $(D_1 \cap H_1)$, F_C corresponds to $(D_1 \cap H_0)$, $T_{\bar{C}}$ corresponds to $(D_0 \cap H_0)$ and $F_{\bar{C}}$ corresponds to $(D_0 \cap H_1)$.

Using the above relations, after a search of a region that has resulted in a contact, the target's position distribution can be defined by:

(31) $P(S_i|C) = P(S_i|T_C) \cdot P(T_C|C) + P(S_i|F_C) \cdot P(F_C|C)$

where $i = 1, 2, \cdots, n$ since $T_C = T_C \cap C$, $F_C = F_C \cap C$ and $P(S_i|C) = [P(S_i \cap T_C) + P(S_i \cap F_C)]/P(C)$. The probability $p = P(T_C|C)$ has been called the credibility of the contact. In terms of p Equation 31 becomes:

(32) $P(S_i|C) = P(S_i|T_C) \cdot p + P(S_i|F_C) \cdot (1 - p)$.

Again using the above relations and $\bar{p} = P(T_{\bar{C}}|\bar{C})$, after a search of a region that has resulted in no contact, the target's position distribution can be defined by:

(33) $P(S_i|\bar{C}) = P(S_i|T_{\bar{C}}) \cdot \bar{p} + P(S_i|F_{\bar{C}}) \cdot (1 - \bar{p})$

where $i = 1, 2, \cdots, n$.

The values of $P(T_C|S_j)$ and of $P(F_{\bar{C}}|S_j) = 1 - P(T_C|S_j)$ are determined by the characteristics of the search in the jth subregion for $j = 1, 2, \cdots, n$. With values for these two probabilities, $P(S_i|T_C) = P(T_C|S_i) \cdot P(S_i)/[\Sigma P(T_C|S_j) \cdot P(S_j)]$ and $P(S_i|F_{\bar{C}}) = P(F_{\bar{C}}|S_i) \cdot P(S_i)/[\Sigma P(F_{\bar{C}}|S_j) \cdot P(S_j)]$.

In one positive information model, p is determined subjectively based on the detection system and the nature of the

search and $P(S_i|F_c) = P(S_i)$. This choice for $P(S_i|F_c)$ could be based on the argument: Given the detection system and the nature of the search, for a search that ends in a false contact, the target was not yet detectable. Consequently, a search that ends in a false contact supplies no information about a target's location. This argument implies that there have been no missed detections during the search.

In a negative information model, in keeping with the above choice for $P(S_i|F_c)$, one might choose $P(S_i|T_{\bar{c}}) = P(S_i)$. This choice for $P(S_i|_{\bar{c}})$ could be based on an argument that is a parallel to the one for the choice above for $P(S_i|F_c)$: Given the detection system and the nature of the search, for a search that ends in a true no contact, the target was not yet detectable. Consequently, a search that ends in a true no contact supplies no information about the target's location. In keeping with the above comment, this argument implies that if false alarms occurred during the search, then they were resolved.

If $p = 1$, then $C = T_c$ (a detection) and, using the above expression for $P(S_i|T_c)$, Equation 32 becomes identical to Equation 30. Note that $1 - p$ corresponds to $P(H_0|D_1)$ and not to $P(D_1|H_0)$ which is p_f.

If $\bar{p} = 0$, then $\bar{C} = F_{\bar{c}}$ (a missed detection) and, using the above expression for $P(S_i|F_{\bar{c}})$, Equation 33 becomes identical to Equation 29.

XV. Search Models and Search Theory

Search theory provides a basis for determining optimal search plans for a target whose motion and location are determined within some bounds. Here, an optimal search plan is one for which the probability of finding a target within a given length of time is a maximum, the expected time to find a target is a minimum given the target is found or a search plan for which some other optimal search criterion is satisfied.

Search theory results are based on models of the search process. To the degree that a search model describes a search process, an optimal search plan for a target that is based on the search model should provide guidance for the development of an operationally feasible search plan. However, because of the limitations of analytical search models, an optimal search plan that is based on an analytical search model may give only initial guidance in this regard. The optimal search plans that are described below illustrate this. The search plans are based on the random search model. Because of this, the requirement on the location of search track segments is not realizable and the time to resolve false alarms is ignored.

Optimal search plans based on search models implemented through a monte carlo simulation are not considered here. However, with sufficient information, such plans have the potential of being both implementable and more optimal in a real sense than an optimal search plan based on an analytical search model.

Three Optimal Search Plans: The three optimal search plans differ in their criterion for an optimal search plan. However, each one is based on the following search model: A target is fixed at some point in a region that consists of n subregions. A search in a subreg on is a random search in the sense of the definition in Section X and a searchers sweep width there is a constant. In addition, a search of a subregion will not detect a target which is in another subregion. To determine a plan, let S_i = (the target is in subregion i) for $i = 1,2, \cdots ,n$ and let $p_i = P(S_i)$ be the prior probability that the target is in the ith subregion. Let W_i be the sweep width in the ith subregion. Let $\delta_i = A_i/W_i$ where A_i is the area of the ith subregion and δ_i is the expected trac] length to find the target by a search of the ith subregion given the target is in the ith subregion, a characteristic length. The probability P that the target will be detected by a random search is given by:

(34) $\qquad P = \Sigma\ [1 - \exp(-l_i/\delta_i)] \cdot p_i$

where the sum index $i = 1,2, \cdots ,n$ and l_i is the track length of the search in the ith subregion.

The first criterion: Choose l_i nat P is a maximum subject to the two constraints: 1. $1 = \Sigma\ l_i$ and 2. $l_i \geq 0$ where the index $i = 1,2, \cdots ,n$. Determining this choice is a nonlinear optimization problem whose solution is given in Reference 20. It is:

(35)
$$l_i^*/\delta_i = \ln(p_i/\delta_i) - L(k) \qquad i = 1,2, \cdots , k$$
$$l_i^*/\delta_i = 0 \qquad\qquad\qquad i = k+1, k+2, \cdots ,n$$

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where $L(k) = (1/\Sigma\delta_j)\cdot\Sigma[\delta_j\cdot\ln(p_j/\delta_j)] - 1/\Sigma\delta_j$ and the sum index $j = 1,2, \cdots ,k$, where the subregions are relabeled so that the following order relation holds: $p_1/\delta_1 > p_2/\delta_2 > \cdots > p_n/\delta_n$ and where k is chosen so that for $k+1$ the solution for l_{k+1} using $L(k+1)$ is either negative or zero.

The second criterion: Choose l_i so that P is a maximum subject to the two constraints: 1. $c = \Sigma c_i$ and 2. $c_i \geq 0$ where the index $i = 1,2, \cdots ,n$, $c_i = k_i\cdot l_i$ is the cost of the search in the ith subregion and k_i is the cost per unit track length in that subregion. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 35 by replacing δ_i by $\epsilon_i = k_i\cdot\delta_i$ and labeling the subregions so that $p_1/\epsilon_i > p_2/\epsilon_i > \cdots > p_n/\epsilon_i$. The basis for this can be seen by replacing l_i/δ_i by its equivalent c_i/ϵ_i in the exponential term in Equation 32.

The third criterion: Choose l_i so that the expected utility of the search is a maximum subject to the two constraints: 1. $l = \Sigma l_i$ and 2. $l_i \geq 0$ where the index $i = 1,2, \cdots ,n$. For this criterion, the solution to the corresponding nonlinear optimization problem can be obtained from Equation 31 by replacing p_i by q_i where $q_i = u_i\cdot p_i$ and u_i is the utility of detecting the target given it is in the ith subregion. And, in addition, labeling the subregions so that $q_1/\delta_1 > q_2/\delta_2 > \cdots > q_n/\delta_n$. The basis for this can be seen by multiplying the summation term in Equation 34 by u_i so that

the resulting equation gives the expected utility of the search given the utility of not detecting the target is zero.

Equation 35 can be used to determine an order of search for the subregions which will effectively minimize the expected track length required to detect a target given it is detected. To do this, divide the available track length l into units small enough so that with a single unit only the 1st subregion would be searched. Then allocate one unit to the search of the 1st subregion. If the search is unsuccessful, determine the optimum allocation for two units. Then search with a second unit so that the first search with the first unit plus the second search with the second unit satisfy the optimum allocation for two units. If the search is unsuccessful, continue in this fashion until either the target is found or all the track length is expended. That this allocation order will effectively minimize the expected track length required to detect a target given it is detected can be argued as follows: Let L be the track length at detection, let l_u be a unit of track length and let n be the number of units. Then the value of the probability $P(L \leq i \cdot l_u)$ that the target will be detected on or before the ith step of the search for the given allocation order will be greater than or equal to its value for any other allocation order with the same allocation step size. Since the value of $P(L \leq l)$ will be equal to its value for any other allocation order of the optimum allocation and since $P(L \leq i \cdot l_u | L \leq l) = P(L \leq i \cdot l_u)/P(L \leq l)$, the value of the distribution function $F_L(i \cdot l_u | L \leq l) = P(L \leq i \cdot l_u | L \leq l)$

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will be greater than or equal to its value for any other allocation order. This implies that the expected track length given detection $E(L|L \leq 1) = \Sigma [1 - F_L(i \cdot l_u | L \leq 1)]$ where the sum index $i = 1, 2, \cdots, n$ is effectively a minimum for the given allocation order. A search based on the optimum allocation given by Equation 35 and the given allocation order is equivalent to the following search: After an allocation of track length l_u and an unsuccessful search, new values for $P(S_i)$ are calculated using Equation 29 and then Equation 35 is used with these new values to determine the next optimum allocation. A discussion of this procedure is given in Reference 6. And an example of its application is given in Reference 21.

Equation 35 also defines an optimal search plan for a detection system that searches beams and can be described by Equation 33 by replacing l_i by t_i where t_i is the time the ith beam is searched and by replacing δ_i by τ_i where τ_i, a characteristic time, is the expected time to detect the target by a search of the ith beam given the target is in the ith beam.

For a more extensive discussion of search theory and its application to military operations research, see Reference 22.

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